

## ON THE STANLEY DEPTH OF POWERS OF EDGE IDEALS

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ABSTRACT. Let  $\mathbb{K}$  be a field and  $S = \mathbb{K}[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables over  $\mathbb{K}$ . Let  $G$  be a graph with  $n$  vertices. Assume that  $I = I(G)$  is the edge ideal of  $G$  and  $p$  is the number of its bipartite connected components. We prove that for every positive integer  $k$ , the inequalities  $\text{sdepth}(I^k/I^{k+1}) \geq p$  and  $\text{sdepth}(S/I^k) \geq p$  hold. As a consequence, we conclude that  $S/I^k$  satisfies the Stanley's inequality for every integer  $k \geq n-1$ . Also, it follows that  $I^k/I^{k+1}$  satisfies the Stanley's inequality for every integer  $k \gg 0$ . Furthermore, we prove that if (i)  $G$  is a non-bipartite graph, or (ii) at least one of the connected components of  $G$  is a tree with at least one edge, then  $I^k$  satisfies the Stanley's inequality for every integer  $k \geq n-1$ . Moreover, we verify a conjecture of the author in special cases.

## 1. INTRODUCTION

Let  $\mathbb{K}$  be a field and let  $S = \mathbb{K}[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables over  $\mathbb{K}$ . Let  $M$  be a finitely generated  $\mathbb{Z}^n$ -graded  $S$ -module. Let  $u \in M$  be a homogeneous element and  $Z \subseteq \{x_1, \dots, x_n\}$ . The  $\mathbb{K}$ -subspace  $u\mathbb{K}[Z]$  generated by all elements  $uv$  with  $v \in \mathbb{K}[Z]$  is called a *Stanley space* of dimension  $|Z|$ , if it is a free  $\mathbb{K}[Z]$ -module. Here, as usual,  $|Z|$  denotes the number of elements of  $Z$ . A decomposition  $\mathcal{D}$  of  $M$  as a finite direct sum of Stanley spaces is called a *Stanley decomposition* of  $M$ . The minimum dimension of a Stanley space in  $\mathcal{D}$  is called the *Stanley depth* of  $\mathcal{D}$  and is denoted by  $\text{sdepth}(\mathcal{D})$ . The quantity

$$\text{sdepth}(M) := \max \{ \text{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M \}$$

is called the *Stanley depth* of  $M$ . We say that a  $\mathbb{Z}^n$ -graded  $S$ -module  $M$  satisfies *Stanley's inequality* if

$$\text{depth}(M) \leq \text{sdepth}(M).$$

In fact, Stanley [16] conjectured that every  $\mathbb{Z}^n$ -graded  $S$ -module satisfies Stanley's inequality. This conjecture has been recently disproved in [1]. However, it is still interesting to find the classes of  $\mathbb{Z}^n$ -graded  $S$ -modules which satisfy Stanley's inequality. For a reader friendly introduction to Stanley depth, we refer to [10] and for a nice survey on this topic, we refer to [6].

Let  $G$  be a graph with vertex set  $V(G) = \{v_1, \dots, v_n\}$ . The edge ideal  $I(G)$  of  $G$  is the ideal of  $S$  generated by the squarefree monomials  $x_i x_j$ , where  $\{v_i, v_j\}$  is an edge of  $G$ . In [12], the authors proved that if  $G$  is a forest (i.e., a graph with no

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cycle), then  $S/I(G)^k$  satisfies the Stanley's inequality for every integer  $K \gg 0$ . Also, it was shown in [2] that  $I(G)^k/I(G)^{k+1}$  satisfies the Stanley's inequality for every forest  $G$  and every integer  $K \gg 0$ . The aim of this paper is to extend these results to the whole class of graphs. In Theorem 2.3, we prove that for every graph  $G$ , the inequality  $\text{sdepth}(S/I(G)^k) \geq p$  holds, where  $p$  is the number of bipartite components of  $G$ . Combining this inequality with a recent result of Trung [17], we conclude that  $S/I(G)^k$  satisfies the Stanley's inequality for every integer  $k \geq n - 1$  (see Corollary 2.5). In Theorem 2.2, we study the Stanley depth of  $I(G)^k/I(G)^{k+1}$  and prove that  $\text{sdepth}(I(G)^k/I(G)^{k+1}) \geq p$ , for every integer  $k \geq 0$ . Combining this inequality with a result of Herzog and Hibi [7], we deduce that  $I(G)^k/I(G)^{k+1}$  satisfies the Stanley's inequality for large  $k$  (see Corollary 2.6).

In section 3, we investigate the Stanley depth of  $I(G)^k$ , for a positive integer  $k$ . In Theorem 3.1, we determine a lower bound for the Stanley depth of  $I(G)^k$ . In Corollaries 3.2 and 3.5, we prove that if (i)  $G$  is a non-bipartite graph, or (ii) at least one of the connected components of  $G$  is a tree (i.e., a connected forest) with at least one edge, then for every positive integer  $k$ , the Stanley depth of  $I(G)^k$  is at least one more than the number of bipartite connected components of  $G$ . Then we conclude that for these classes of graphs, the ideal  $I(G)^k$  satisfies the Stanley's inequality, for every  $k \geq n - 1$ , where  $n = |V(G)|$  (see Corollary 3.6).

## 2. STANLEY DEPTH OF QUOTIENT OF POWERS OF EDGE IDEALS

In this section, we study the Stanley depth of quotient of powers of edge ideals. Before starting the proofs, we remind that for every graph  $G$  and every subset  $W$  of  $V(G)$ , the graph  $G \setminus W$  is the graph formed by removing the vertices of  $W$  from the vertex set of  $G$  and deleting any edge in  $G$  that contains a vertex of  $W$ .

The first main result of this paper asserts that the number of bipartite connected components of  $G$  is a lower bound for the Stanley depth of  $I(G)^k/I(G)^{k+1}$ , for every nonnegative integer  $k$ . We first need the following lemma.

**Lemma 2.1.** *Let  $G$  be a connected bipartite graph with edge ideal  $I = I(G)$ . Then for every integer  $k \geq 0$ , we have  $\text{sdepth}(I^k/I^{k+1}) \geq 1$ .*

*Proof.* By [4, Proposition 2.13], it is enough to prove that for every integer  $k \geq 0$ , we have  $\text{depth}(I^k/I^{k+1}) \geq 1$ . We use induction on  $k$ . For  $k = 0$ , the assertion says that  $\text{depth}(S/I) \geq 1$  which is trivial. Thus, assume that  $k \geq 1$ . Consider the following short exact sequence:

$$0 \longrightarrow I^k/I^{k+1} \longrightarrow S/I^{k+1} \longrightarrow S/I^k \longrightarrow 0.$$

It follows from depth lemma that

$$\text{depth}(I^k/I^{k+1}) \geq \min\{\text{depth}(S/I^{k+1}), \text{depth}(S/I^k) + 1\} \geq 1,$$

where the last inequality follows from [9, Lemma 2.6].  $\square$

We are now ready to prove the first main result of this paper.

**Theorem 2.2.** *Let  $G$  be a graph with edge ideal  $I = I(G)$ . Suppose that  $p$  is the number of bipartite connected components of  $G$ . Then for every integer  $k \geq 0$ , we have  $\text{sdepth}(I^k/I^{k+1}) \geq p$ .*

*Proof.* Using [8, Lemma 3.6], we may assume that  $G$  has no isolated vertex. We prove the theorem by induction on  $p$ . There is nothing to prove if  $p = 0$ . Thus, assume that  $p \geq 1$  and the assertion is true for every graph with at most  $p - 1$  bipartite connected components. If  $G$  is connected, it follows from  $p \geq 1$  that  $G$  is a bipartite graph and the claim follows from Lemma 2.1. Therefore, assume that  $G$  has at least two connected components. Suppose that  $G_1$  is a bipartite connected component of  $G$ . Let  $L$  and  $J$  be the edge ideals of  $G_1$  and  $G \setminus V(G_1)$ , respectively (we consider  $L$  and  $J$  as ideals in  $S$ ). Then  $I = L + J$ . Therefore,  $I^k = \sum_{s+t=k} L^s J^t$  and thus,

$$(1) \quad I^k/I^{k+1} = \sum_{s+t=k} (L^s J^t + I^{k+1}/I^{k+1}).$$

On the other hand, for every distinct pairs  $(s, t) \neq (l, m)$  of nonnegative integers with  $s + t = l + m = k$ , we have

$$\begin{aligned} L^s J^t \cap L^l J^m &= (L^s \cap J^t) \cap (L^l \cap J^m) = L^{\max\{s,l\}} \cap J^{\max\{t,m\}} \\ &= L^{\max\{s,l\}} J^{\max\{t,m\}} \subseteq (L + J)^{k+1} = I^{k+1}. \end{aligned}$$

This shows that the sum in (1) is direct and therefore by the definition of Stanley depth we have

$$\text{sdepth}(I^k/I^{k+1}) \geq \min_{s+t=k} \{\text{sdepth}(L^s J^t + I^{k+1}/I^{k+1})\}.$$

Hence, it is enough to show that for every pair  $(s, t)$  of nonnegative integers with  $s + t = k$ ,

$$\text{sdepth}(L^s J^t + I^{k+1}/I^{k+1}) \geq p,$$

or equivalently

$$\text{sdepth}(L^s J^t / L^s J^t \cap I^{k+1}) \geq p.$$

Note that for every pair  $(s, t)$  of nonnegative integers with  $s + t = k$ ,

$$\begin{aligned} L^{s+1} J^t + L^s J^{t+1} &\subseteq L^s J^t \cap I^{k+1} = L^s J^t \cap (L + J)^{k+1} \subseteq L^s \cap J^t \cap (L^{s+1} + J^{t+1}) = \\ &= (L^s \cap J^t \cap L^{s+1}) + (L^s \cap J^t \cap J^{t+1}) = (J^t \cap L^{s+1}) + (L^s \cap J^{t+1}) \\ &= L^{s+1} J^t + L^s J^{t+1}. \end{aligned}$$

This shows that

$$L^{s+1} J^t + L^s J^{t+1} = L^s J^t \cap I^{k+1}$$

and thus,

$$L^s J^t / L^s J^t \cap I^{k+1} = L^s J^t / (L^{s+1} J^t + L^s J^{t+1}).$$

Set  $S' = \mathbb{K}[x_i \mid v_i \in V(G_1)]$  and  $S'' = \mathbb{K}[x_i \mid v_i \notin V(G_1)]$ . Also, set  $L' = L \cap S'$  and  $J' = J \cap S''$ . By Lemma 2.1  $\text{sdepth}_{S'}(L'/L'^{s+1}) \geq 1$  and by induction hypothesis

$\text{sdepth}_{S''}(J^t/J^{t+1}) \geq p-1$ . Hence, there exist Stanley decompositions

$$\mathcal{D} : L^s/J^{s+1} = \bigoplus_{i=1}^r u_i \mathbb{K}[Z_i] \quad \text{and} \quad \mathcal{D}' : J^t/J^{t+1} = \bigoplus_{j=1}^{r'} u'_j \mathbb{K}[Z'_j]$$

with  $\text{sdepth}(\mathcal{D}) \geq 1$  and  $\text{sdepth}(\mathcal{D}') \geq p-1$ . One can easily check that

$$L^s J^t / (L^{s+1} J^t + L^s J^{t+1}) = \bigoplus_{i=1}^r \bigoplus_{j=1}^{r'} u_i u'_j \mathbb{K}[Z_i \cup Z'_j]$$

is a Stanley decomposition and since for every pair of integers  $i$  and  $j$  with  $1 \leq i \leq r$  and  $1 \leq j \leq r'$ , we have  $Z_i \cap Z'_j = \emptyset$ , it follows that

$$\text{sdepth}(L^s J^t / L^s J^t \cap I^{k+1}) = \text{sdepth}(L^s J^t / (L^{s+1} J^t + L^s J^{t+1})) \geq p$$

and this completes the proof.  $\square$

The following theorem is a consequence of Theorem 2.2. It shows that the number of bipartite connected components of  $G$  is a lower bound for the Stanley depth of  $S/I(G)^k$ , for every positive integer  $k$ .

**Theorem 2.3.** *Let  $G$  be a graph with edge ideal  $I = I(G)$ . Suppose that  $p$  is the number of bipartite connected components of  $G$ . Then for every integer  $k \geq 1$ , we have  $\text{sdepth}(S/I^k) \geq p$ .*

*Proof.* We use induction on  $k$ . For  $k = 1$ , it follows from Theorem 2.2 that  $\text{sdepth}(S/I) = \text{sdepth}(I^0/I) \geq p$ . Thus, assume that  $k \geq 2$  and  $\text{sdepth}(S/I^{k-1}) \geq p$ . Consider the following short exact sequence:

$$0 \longrightarrow I^{k-1}/I^k \longrightarrow S/I^k \longrightarrow S/I^{k-1} \longrightarrow 0.$$

Using [13, Lemma 2.2], we conclude that

$$\text{sdepth}(S/I^k) \geq \min\{\text{sdepth}(I^{k-1}/I^k), \text{sdepth}(S/I^{k-1})\}.$$

Now, Theorem 2.2 and the induction hypothesis imply that  $\text{sdepth}(S/I^k) \geq p$ .  $\square$

Let  $\mathbb{K}$  be a field and  $S = \mathbb{K}[x_1, x_2, \dots, x_n]$  be the polynomial ring in  $n$  variables over the field  $\mathbb{K}$ , and let  $I \subset S$  be a monomial ideal. A classical result by Burch [5] states that

$$\min_k \text{depth}(S/I^k) \leq n - \ell(I),$$

where  $\ell(I)$  is the analytic spread of  $I$ , that is, the dimension of  $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ , where  $\mathcal{R}(I) = \bigoplus_{n=0}^{\infty} I^n = S[It] \subseteq S[t]$  is the Rees ring of  $I$  and  $\mathfrak{m} = (x_1, \dots, x_n)$  is the maximal ideal of  $S$ . By a theorem of Brodmann [3],  $\text{depth}(S/I^k)$  is constant for large  $k$ . We call this constant value the *limit depth* of  $I$ , and denote it by  $\lim_{t \rightarrow \infty} \text{depth}(S/I^t)$ . Brodmann improved the Burch's inequality by showing that

$$\lim_{k \rightarrow \infty} \text{depth}(S/I^k) \leq n - \ell(I).$$

Let  $G$  be a graph with edge ideal  $I = I(G)$ . Suppose that  $n$  is the number of vertices of  $G$  and  $p$  is the number of its bipartite connected components. It follows from [18, Page 50] that  $\ell(I) = n - p$ . Thus, using the Burch's inequality, we conclude that

$$\lim_{k \rightarrow \infty} \text{depth}(S/I^k) \leq p.$$

Recently, Trung [17] proved that we have in fact equality in the above inequality. Indeed, he proved the following stronger result.

**Theorem 2.4.** ([17, Theorems 4.4 and 4.6]) *Let  $G$  be a graph with edge ideal  $I = I(G)$ . Suppose that  $n$  is the number of vertices of  $G$  and  $p$  is the number of its bipartite connected components. Then for every integer  $k \geq n - 1$ , we have  $\text{depth}(S/I^k) = p$ .*

In [12, Corollary 2.8], the authors proved that  $S/I^k$  satisfies the Stanley's inequality for every  $k \gg 0$ , when  $I$  is the edge ideal of a forest. The following corollary is an extension of this result and it is an immediate consequence of Theorems 2.3 and 2.4.

**Corollary 2.5.** *Let  $G$  be a graph with edge ideal  $I = I(G)$ . Suppose that  $n$  is the number of vertices of  $G$ . Then  $S/I^k$  satisfy the Stanley's inequality, for every integer  $k \geq n - 1$ .*

In [2, Corollary 3.2], the authors proved that  $I^k/I^{k+1}$  satisfies the Stanley's inequality for every  $k \gg 0$ , when  $I$  is the edge ideal of a forest. The following corollary is an extension of this result and shows that  $I^k/I^{k+1}$  satisfies the Stanley's inequality for every edge ideal  $I$  and every integer  $k \gg 0$ . Unfortunately, we are not able to determine an upper bound for the least integer  $k$ , such that  $I^k/I^{k+1}$  satisfies the Stanley's inequality.

**Corollary 2.6.** *Let  $G$  be a graph with edge ideal  $I = I(G)$ . Then  $I^k/I^{k+1}$  satisfies the Stanley's inequality, for every integer  $k \gg 0$ .*

*Proof.* It follows from [7, Theorem 1.2] and [17, Theorem 4.4] that

$$\lim_{k \rightarrow \infty} \text{depth}(I^k/I^{k+1}) = \lim_{k \rightarrow \infty} \text{depth}(S/I^k) = p.$$

Now, Theorem 2.2 completes the proof.  $\square$

Let  $I \subset S$  be an arbitrary ideal. An element  $f \in S$  is *integral* over  $I$ , if there exists an equation

$$f^k + c_1 f^{k-1} + \dots + c_{k-1} f + c_k = 0 \quad \text{with } c_i \in I^i.$$

The set of elements  $\bar{I}$  in  $S$  which are integral over  $I$  is the *integral closure* of  $I$ . The ideal  $I$  is *integrally closed*, if  $I = \bar{I}$ .

In [14], the author proposed the following conjecture regarding the Stanley depth of integrally closed monomial ideals.

**Conjecture 2.7.** ([14, Conjecture 2.6]) *Let  $I \subset S$  be an integrally closed monomial ideal. Then  $\text{sdepth}(S/I) \geq n - \ell(I)$  and  $\text{sdepth}(I) \geq n - \ell(I) + 1$ .*

This conjecture is known to be true for some classes of monomial ideals. For example, it is shown in [11, Theorem 2.5] that the conjecture is true for every weakly polymatroidal ideal which is generated in a single degree. Also, in [15, Corollary 3.4], the author proved the conjecture for every squarefree monomial ideal which is generated in a single degree. Now, Theorem 2.3 shows that the conjectured inequality for  $S/I$  is true, when  $I$  is a power of the edge ideal of a graph.

### 3. STANLEY DEPTH OF POWERS OF EDGE IDEALS

In this section we determine a lower bound for the Stanley depth of powers of edge ideals. In particular, we show that if either  $G$  is a non-bipartite graph, or has a connected component which is a tree with at least one edge, then for every positive integer  $k$ , the Stanley depth of  $I(G)^k$  is at least one more than the number of the bipartite connected components of  $G$ . We conclude that in these cases  $I(G)^k$  satisfies the Stanley's inequality for every  $k \geq |V(G)| - 1$ .

**Theorem 3.1.** *Let  $G$  be a graph with edge ideal  $I = I(G)$ . Assume that  $H$  is a connected component of  $G$  with at least one edge. Suppose that  $h$  is the number of bipartite connected components of  $G \setminus V(H)$ . Then for every integer  $k \geq 1$ , we have*

$$\text{sdepth}(I^k) \geq \min_{1 \leq l \leq k} \{\text{sdepth}_{S'}(I(H)^l)\} + h,$$

where  $S' = \mathbb{K}[x_i \mid v_i \in V(H)]$ .

*Proof.* Using [8, Lemma 3.6], we may assume that  $G \setminus V(H)$  has no isolated vertex. We prove the theorem by induction on the number of connected components of  $G$ . If  $G$  is connected, the  $G = H$  and there is nothing to prove. Thus, assume that  $G$  is not a connected graph. Set  $S'' = \mathbb{K}[x_i \mid v_i \notin V(H)]$ . Let  $L' \subset S'$  and  $J' \subset S''$  be the edge ideals of  $H$  and  $G \setminus V(H)$ , respectively. Set  $L = L'S$  and  $J = J'S$ . Then  $I = L + J$ . Therefore,  $I^k = \sum_{t=0}^k L^t J^{k-t}$ . For every integer  $l$  with  $1 \leq l \leq k$ , we have the following short exact sequence:

$$0 \longrightarrow \sum_{t=0}^{l-1} L^t J^{k-t} \longrightarrow \sum_{t=0}^l L^t J^{k-t} \longrightarrow \sum_{t=0}^l L^t J^{k-t} / \sum_{t=0}^{l-1} L^t J^{k-t} \longrightarrow 0.$$

Using [13, Lemma 2.2], we conclude that

$$\text{sdepth}\left(\sum_{t=0}^l L^t J^{k-t}\right) \geq \min\left\{\text{sdepth}\left(\sum_{t=0}^{l-1} L^t J^{k-t}\right), \text{sdepth}\left(\sum_{t=0}^l L^t J^{k-t} / \sum_{t=0}^{l-1} L^t J^{k-t}\right)\right\}.$$

This implies that

$$\begin{aligned} \text{sdepth}(I^k) &= \text{sdepth}\left(\sum_{t=0}^k L^t J^{k-t}\right) \\ &\geq \min\left\{\text{sdepth}(J^k), \text{sdepth}\left(\sum_{t=0}^l L^t J^{k-t} / \sum_{t=0}^{l-1} L^t J^{k-t}\right) : l = 1, \dots, k\right\}. \end{aligned}$$

Let  $H'$  be a connected component of  $G \setminus V(H)$  and set  $T = \mathbb{K}[x_i \mid v_i \in V(H')]$ . Notice that  $G \setminus (V(H) \cup V(H'))$  has at least  $h-1$  bipartite connected components. Also, note that  $H'$  has at least one edge and by [6, Corollary 24], we have  $\text{sdepth}_T(I(H')^l) \geq 1$ , for every positive integer  $l$ . Therefore, using induction hypothesis and [8, Lemma 3.6], we conclude that

$$\begin{aligned} \text{sdepth}(J^k) &= \text{sdepth}_{S''}(J'^k) + |V(H)| \geq \min_{1 \leq l \leq k} \{\text{sdepth}_T(I(H')^l)\} + (h-1) + |V(H)| \\ &\geq 1 + (h-1) + \min_{1 \leq l \leq k} \{\text{sdepth}_{S'}(I(H)^l)\} = \min_{1 \leq l \leq k} \{\text{sdepth}_{S'}(I(H)^l)\} + h. \end{aligned}$$

Therefore, it is enough to show that for every integer  $l$  with  $1 \leq l \leq k$ , the inequality

$$\text{sdepth}\left(\sum_{t=0}^l L^t J^{k-t} / \sum_{t=0}^{l-1} L^t J^{k-t}\right) \geq \min_{1 \leq l \leq k} \{\text{sdepth}_{S'}(I(H)^l)\} + h$$

holds. Notice that

$$\sum_{t=0}^l L^t J^{k-t} / \sum_{t=0}^{l-1} L^t J^{k-t} \cong L^l J^{k-l} / (L^l J^{k-l} \cap \sum_{t=0}^{l-1} L^t J^{k-t})$$

and

$$\begin{aligned} L^l J^{k-l} \cap \sum_{t=0}^{l-1} L^t J^{k-t} &= \sum_{t=0}^{l-1} (L^l J^{k-l} \cap L^t J^{k-t}) = \sum_{t=0}^{l-1} (L^l \cap J^{k-l} \cap L^t \cap J^{k-t}) = \\ \sum_{t=0}^{l-1} (L^l \cap J^{k-t}) &= \sum_{t=0}^{l-1} L^l J^{k-t} = L^l \sum_{t=0}^{l-1} J^{k-t} = L^l J^{k-l+1}. \end{aligned}$$

Hence,

$$(2) \quad \sum_{t=0}^l L^t J^{k-t} / \sum_{t=0}^{l-1} L^t J^{k-t} \cong L^l J^{k-l} / L^l J^{k-l+1}.$$

By Theorem 2.2  $\text{sdepth}_{S''}(J'^{k-l} / J'^{k-l+1}) \geq h$ . Consider the Stanley decompositions

$$\mathcal{D} : L^l = \bigoplus_{i=1}^r u_i \mathbb{K}[Z_i] \quad \text{and} \quad \mathcal{D}' : J'^{k-l} / J'^{k-l+1} = \bigoplus_{j=1}^{r'} u'_j \mathbb{K}[Z'_j]$$

with  $\text{sdepth}(\mathcal{D}) = \text{sdepth}(L^l)$  and  $\text{sdepth}(\mathcal{D}') \geq h$ . One can easily check that

$$L^l J^{k-l} / L^l J^{k-l+1} = \bigoplus_{i=1}^r \bigoplus_{j=1}^{r'} u_i u'_j \mathbb{K}[Z_i \cup Z'_j]$$

is a Stanley decomposition and since for every pair of integers  $i$  and  $j$  with  $1 \leq i \leq r$  and  $1 \leq j \leq r'$ , we have  $Z_i \cap Z'_j = \emptyset$ , it follows that

$$\text{sdepth}(L^l J^{k-l} / L^l J^{k-l+1}) \geq \text{sdepth}(L^l) + h \geq \min_{1 \leq l \leq k} \{\text{sdepth}_{S'}(I(H)^l)\} + h$$

and thus, the isomorphism (2) completes the proof.  $\square$



The following corollary shows that if  $G$  has a non-bipartite connected component, then for every positive integer  $k$ , the Stanley depth of  $I(G)^k$  is at least one more than the number of bipartite connected components of  $G$ .

**Corollary 3.2.** *Let  $G$  be a non-bipartite graph with edge ideal  $I = I(G)$ . Suppose that  $p$  is the number of bipartite connected components of  $G$ . Then for every integer  $k \geq 1$ , we have  $\text{sdepth}(I^k) \geq p + 1$ .*

*Proof.* Note that  $G$  has a non-bipartite connected component, say  $H$ . Thus, the assertion follows by applying Theorem 3.1 and [6, Corollary 24].  $\square$

In view of Corollary 3.2, we expect that the inequality  $\text{sdepth}(I(G)^k) \geq p + 1$  can be true for every graph  $G$  with  $p$  bipartite connected components and for every positive integer  $k$ . In order to prove this inequality one only needs to prove it when  $G$  is a connected bipartite graph (with at least one edge). Then the desired inequality follows from Theorem 3.1. Thus, one can ask the following question.

**Question 3.3.** *Let  $G$  be a connected bipartite graph (with at least one edge) and suppose  $k \geq 1$  is an integer. Is it true that  $\text{sdepth}(I(G)^k) \geq 2$ ?*

By [15, Corollary 3.4] and [18, Page 50], we know that the answer of Question 3.3 is positive for  $k = 1$ . Unfortunately, we are not able to give a complete answer to Question 3.3. However, we give a positive answer to this question, when  $G$  is a tree. We first need to introduce some basic notions from graph theory.

Let  $G$  be a graph with vertex set  $V(G) = \{v_1, \dots, v_n\}$ . For a vertex  $v_i$ , the *neighbor set* of  $v_i$  is  $N(v_i) = \{v_j \mid \{v_i, v_j\} \text{ is an edge of } G\}$ . The vertex  $v_i$  is called a *leaf* if  $N(v_i)$  is a singleton.

**Proposition 3.4.** *Let  $G$  be a tree with at least one edge. Then for every integer  $k \geq 1$ , we have  $\text{sdepth}(I(G)^k) \geq 2$ .*

*Proof.* Set  $I = I(G)$ . We prove the claim by induction on  $n + k$ , where  $n$  is the number of vertices of  $G$ . If  $k = 1$ , then the result follows from [15, Corollary 3.4] and [18, Page 50]. If  $n = 2$ , then  $I^k$  is a principal ideal and the assertion is trivially true. We therefore suppose that  $k \geq 2$  and  $n \geq 3$ . Let  $v_1$  be a leaf of  $G$  and assume that  $N(v_1) = \{v_2\}$ . Let  $S' = \mathbb{K}[x_2, \dots, x_n]$  be the polynomial ring obtained from  $S$  by deleting the variable  $x_1$ . Then

$$I^k = (I^k \cap S') \bigoplus x_1(I^k : x_1)$$

and therefore by definition of the Stanley depth it is enough to prove that

- (i)  $\text{sdepth}_{S'}(I^k \cap S') \geq 2$ .
- (ii)  $\text{sdepth}_S(I^k : x_1) \geq 2$ .

To prove (i), let  $I' \subseteq S'$  be the edge ideal of  $G \setminus \{v_1\}$ . Then  $I^k \cap S' = I'^k$ . Since  $G \setminus \{v_1\}$  is a tree with  $n - 1$  vertices, the induction hypothesis implies that

$$\text{sdepth}_{S'}(I^k \cap S') \geq 2.$$



Next, we show that  $\text{sdepth}_S(I^k : x_1) \geq 2$ .

Let  $S'' = \mathbb{K}[x_1, x_3, \dots, x_n]$  be the polynomial ring obtained from  $S$  by deleting the variable  $x_2$ . Since

$$(I^k : x_1) = ((I^k : x_1) \cap S'') \bigoplus x_2(I^k : x_1x_2),$$

by [9, Lemma 2.10], we conclude that

$$(I^k : x_1) = ((I^k : x_1) \cap S'') \bigoplus x_2 I^{k-1}.$$

Using the induction hypothesis, it follows that  $\text{sdepth}(I^{k-1}) \geq 2$ . Thus, to complete the proof we should show that if  $(I^k : x_1) \cap S'' \neq 0$ , then

$$\text{sdepth}_{S''}((I^k : x_1) \cap S'') \geq 2.$$

Thus assume that  $(I^k : x_1) \cap S'' \neq 0$ . Set  $G' = G \setminus \{x_1, x_2\}$ . Since  $N(x_1) = \{x_2\}$ , it follows

$$(I^k : x_1) \cap S'' = I(G')^k S''.$$

Set  $T = \mathbb{K}[x_3, \dots, x_n]$ . Now, [8, Lemma 3.6] and [6, Corollary 24] imply that

$$\text{sdepth}_{S''}((I^k : x_1) \cap S'') = \text{sdepth}_{S''}(I(G')^k S'') = \text{sdepth}_T(I(G')^k) + 1 \geq 2.$$

□

The following Corollary is a consequence of Theorem 3.1 and Proposition 3.4.

**Corollary 3.5.** *Let  $G$  be a graph with edge ideal  $I = I(G)$ . Assume that  $G$  has  $p$  bipartite connected components and suppose that at least one of the connected components of  $G$  is a tree with at least one edge. Then for every integer  $k \geq 1$ , we have  $\text{sdepth}(I^k) \geq p + 1$ .*

*Proof.* Apply Theorem 3.1 by assuming that  $H$  is the connected component of  $G$  which is a tree with at least one edge. Note that by Proposition 3.4, for every integer  $k \geq 1$ ,

$$\min_{1 \leq l \leq k} \{\text{sdepth}_{S'}(I(H)^l)\} \geq 2,$$

where  $S' = \mathbb{K}[x_i \mid v_i \in V(H)]$ . □

As an immediate consequence of Theorem 2.4, Corollaries 3.2 and 3.5, we obtain the following result.

**Corollary 3.6.** *Assume that  $G$  is a graph with  $n$  vertices, such that*

- (i)  *$G$  is a non-bipartite graph, or*
- (ii) *at least one of the connected components of  $G$  is a tree with at least one edge.*

*Then for every integer  $k \geq n - 1$ , the ideal  $I(G)^k$  satisfies the Stanley's inequality.*

We remind that Conjecture 2.7 predicts that  $n - \ell(I) + 1$  is a lower bound for the Stanley depth of  $I$ , when  $I$  is an integrally closed monomial ideal of  $S$ . Note that by Corollaries 3.2 and 3.5, the conjectured inequality is true, when  $I$  is a power of the edge ideal of those graphs which belong to the classes (i) and (ii) of the above corollary.

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